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APPLICATION OF THE CENTRE IMPLICIT METHOD FOR INVESTIGATION OF PRESSURE TRANSIENTS IN PIPELINES

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SUMMARY

This paper investigates the application of the centre implicit method for the determination of the pressure transient in a pipeline, and compares the results with those obtained using the method of characteristics and an experimental investigation. The study shows that there are unique values for the stability criterion (ratio of the linear and time increments) and the artificial viscosity term (a damping factor) used in the numerical computation. The time step and the number of nodes required for the accuracy of the method have been considered. The centre implicit method can be readily adapted to transient flow with variable wave speed provided the established conditions are used.

KEY WORDS Computational methods Pipelines Pressure Transients Waterhammer

INTRODUCTION

Initially, pressure transients in pipelines were analysed using graphical techniques[1] and the friction effect was assumed to be a localized phenomenon at the ends of the pipeline. The accuracy of the graphical method could be improved by increasing the number of localized friction losses, but this approach tends to increase the complexity of the solution and lacked the mathematical rigour. Later, digital computers were introduced for solving the pressure transient equations in pipelines.[2–4] The most commonly used analysis is the method of characteristics (MOC), which has been applied for solving the pressure transient problems in pipelines with various boundary conditions. In the MOC, partial differential equations of hyperbolic type, describing the transient flow, are transformed into ordinary differential equations and solved by a finite difference method. For one-dimensional, single-phase turbulent flow in a pipeline, the MOC predicts, fairly accurately, the pressure transients.[5,6] However, when the MOC is applied to pipe flows at low Reynolds numbers where the flow is two-dimensional, or to multi-phase flow, considerable discrepancies are seen between the predicted and experimental results.[3,7]

The centre implicit method (CIM) was used in the study of pressure transients in pipelines and found to be faster than the MOC when the restriction on the time step is not imposed, but the CIM yielded unsatisfactory results for very sudden and sharp transients.[4] In the CIM, the partial

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differential equations of the hyperbolic type derived for transient flow in pipelines are directly replaced by difference quotients and numerically solved. This procedure could be easily extended to equations satisfying the two-dimensional and/or the multi-phase flow.[8] The paper investigates the CIM for solving pressure transients in pipelines, and the attention is focused on the stability criteria, the artificial viscosity term,[9] a damping factor and the accuracy; and the computer times for the MOC and the CIM are compared.

MATHEMATICAL MODELLING

To investigate the application of CIM for solving the pressure transient equations, a simple pipeline system (Figure 1) with a constant cross-section is considered. The continuity and momentum equations for single-phase one-dimensional flow[6] are

Continuity

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \rho c^2\frac{\partial u}{\partial x} = 0. \tag{1}$$

Momentum

$$\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\mathrm{d}z}{\mathrm{d}x} + 2\frac{f}{D}u|u| = 0, \tag{2}$$

where c is the wave velocity, D the diameter of the pipe, f the friction coefficient, g the gravitational constant, p the transient pressure, t the time, u the mean flow velocity, x the length along the pipe, z the elevation of the pipe and ρ the density of flow.

Method of characteristics

Solving of equations (1) and (2) by the MOC[6] leads to the following ordinary differential equations:

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{1}{\rho c}\frac{\mathrm{d}p}{\mathrm{d}t} + g\frac{\mathrm{d}z}{\mathrm{d}x} + \frac{2f}{D}u|u| = 0, \tag{3}$$

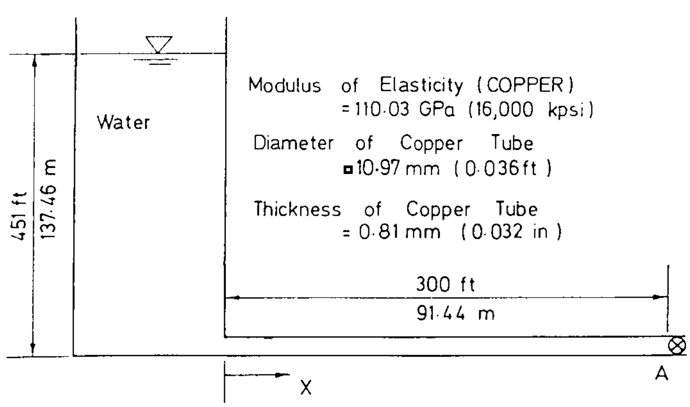

Figure 1. Details of the pipeline system used in the investigation

$$\frac{dx}{dt} = u \pm c. \tag{4}$$

The above equations may be solved by either a graphical or a finite difference method for various boundary conditions. In this study a computer program written in Fortran IV is used for the theoretical prediction of results. Figure 2 shows the theoretical pressure transient curve versus time at the value for the pipeline system in Figure 1, with and without friction in the pipeline for the same rate of flow. The MOC is reported to be unconditionally stable for all mesh sizes Δx, as long as the condition $\Delta t < \Delta x/c$ is satisfied,[4] which has been found to be true.

Centre implicit method

Application of the centre implicit finite difference approach to equations (1) and (2) depends on the first-order Taylor's approximations. For example, if subscripts i and $i+1$ represent the nodes of a grid matrix in a domain of interest, as shown in Figure 3, then the linear gradient of pressure or

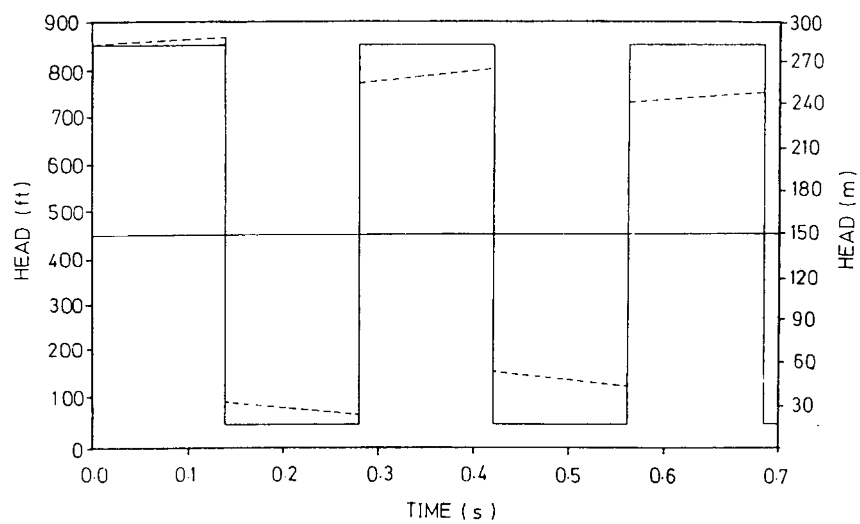

Figure 2. Pressure transients with and without friction for the pipeline of Figure 1 using the method of characteristics

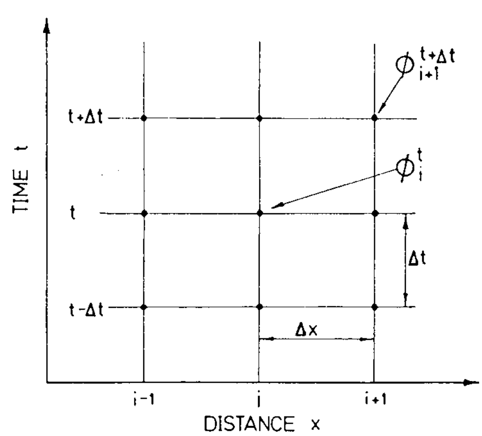

Figure 3. Notation for the centre implicit method

mean velocity, as a variable ϕ, can be expressed as follows:

$$\frac{\partial \phi}{\partial x} = \frac{(2-\theta)(\phi_{i+1}^{t} - \phi_i^t) + \theta(\phi_{i+1}^{t+\Delta t} - \phi_i^{t+\Delta t})}{2\Delta x}, \tag{5}$$

where Δx and Δt represent the linear and time steps, respectively; θ is the artificial viscosity term, which is dimensionless. The constants $(2-\theta)$ and θ are introduced in the above equations as damping factors which are similar to the artificial friction factor cited by other researchers.[4,9] The damping factor is introduced to investigate the amount of overshoot when the transient pressure rises and falls due to sudden closure of a value in the pipeline. If $\theta = 1$, then equation (5) reduces to the standard finite difference form, without any damping factor.

Similarly, the time gradient of pressure or mean velocity, as a variable ϕ, at nodal points can be expressed as follows:

$$\frac{\partial \phi}{\partial t} = \frac{(\phi_i^{t+\Delta t} - \phi_i^t) + (\phi_{i+1}^{t+\Delta t} - \phi_{i+1}^t)}{2\Delta t}. \tag{6}$$

The mean velocity u is the arithmetic average of the nodal mean velocity at the current time step, which is

$$u = 1/2(u_i^t + u_{i+1}^t). \tag{7}$$

Substituting the finite difference equations (5) to (7) into continuity equation (1) and rearranging, yields the following equations:

$$F_1 p_i^{t+\Delta t} + F_2 p_{i+1}^{t+\Delta t} + F_3 u_i^{t+\Delta t} + F_4 u_{i+1}^{t+\Delta t} = F_5, \tag{8}$$

where

$$F_1 = \frac{1}{\rho c^2}\frac{\Delta x}{\Delta t} - \frac{\theta}{2\rho c^2}(u_i^t + u_{i+1}^t), \tag{9a}$$

$$F_2 = \frac{1}{\rho c^2}\frac{\Delta x}{\Delta t} + \frac{\theta}{2\rho c^2}(u_i^t + u_{i+1}^t), \tag{9b}$$

$$F_3 = -\theta, \tag{9c}$$

$$F_4 = \theta, \tag{9d}$$

$$F_5 = \frac{1}{\rho c^2}\frac{\Delta x}{\Delta t}(p_i^t + p_{i+1}^t) + \frac{1}{2\rho c^2}(u_i^t + u_{i+1}^t)(p_i^t - p_{i+1}^t) + (2-\theta)(u_i^t - u_{i+1}^t). \tag{9e}$$

Substituting, the finite difference equations (5)–(7) into the momentum equation (2) and rearranging, yields the following equation:

$$G_1 p_i^{t+\Delta t} + G_2 p_{i+1}^{t+\Delta t} + G_3 u_i^{t+\Delta t} + G_4 u_{i+1}^{t+\Delta t} = G_5 \tag{10}$$

where

$$G_1 = -\frac{\theta}{\rho}, \tag{11a}$$

$$G_2 = \frac{\theta}{\rho}, \tag{11b}$$

$$G_3 = \frac{\Delta x}{\Delta t} - \frac{\theta}{2}(u_i^t + u_{i+1}^t), \tag{11c}$$

$$G_4 = \frac{\Delta x}{\Delta t} + \frac{\theta}{2}(u_i^t + u_{i+1}^t), \tag{11d}$$

$$G_5 = \frac{\Delta x}{\Delta t}(u_i^t + u_{i+1}^t) + \tfrac{1}{2}(2-\theta)(u_i^t + u_{i+1}^t)(u_i^t - u_{i+1}^t)$$
$$+ \frac{(2-\theta)}{\rho}(p_i^t - p_{i+1}^t) - 4\frac{f}{D}u|u|\Delta x - 2g\Delta z. \tag{11e}$$

Equations (8) and (10) are finite difference equations describing the local variations in p and u at any instant of time at a cross-section of the pipe. As there are four unknowns, $p_i^{t+\Delta t}, p_{i+1}^{t+\Delta t}, u_i^{t+\Delta t}$ and $u_{i+1}^{t+\Delta t}$, the solution to these equations at a time step interval needs four equations which are obtained by considering two adjacent nodes. Therefore, in a piping system containing N nodes, the total number of equations that could be written is $(2N-2)$. The remaining two equations are obtained by considering the boundary conditions at the extreme ends of the pipeline. The solution of the $2N$ equations simultaneously is termed simply the centre implicit method, because the gradient functions (i.e. equations (5) and (6)) are taken at the centres of the nodes in both the spatial and time directions.

Method of evaluation

The CIM is examined by applying the numerical method to a simple water hammer analysis problem, where a single phase fluid is carried in a horizontal pipeline of constant cross-section and thickness. The details shown in Figure 1 are identical to the configuration used by Streeter and Lai.[3] The boundary conditions considered are that the upstream pressure remains constant and the pressure transient is caused by a sudden closure of a value at the downstream end. The stability criterion is investigated by considering the effect of the ratios of the linear and time steps, and the artificial viscosity term θ on the pressure transients. Further evaluations are made by comparing the theoretical predictions with the CIM and the MOC, and both with the experimental results presented by Streeter and Lai.[3]

A computer program for solving the finite difference equations has been written in Fortran IV and the flow chart of the program is given in Figure 4. The initial steady conditions are used as the starting values, and the transient pressure p and u at the next time step $t + \Delta t$ are computed by first applying equations (8) and (10) to all nodes. Then the simultaneous equations are assembled to give a matrix of the form $\mathbf{AX} = \mathbf{B}$. The matrix $\mathbf{A}$ is assembled from the coefficients F_1 to F_4 and G_1 to G_4, and the matrix $\mathbf{B}$ is assembled from the coefficients F_5 and G_5. The local friction factor is calculated based on the mean velocity given by equation (7) and the following equations based on Moody's diagram for smooth pipes:

$$f = \frac{16}{Re}; \quad \text{for } Re < 3000 \tag{12a}$$

$$\frac{1}{\sqrt{f}} = 4\log_{10}[(Re)2\sqrt{f}] - 1{\cdot}6; \quad Re \geqslant 3000 \tag{12b}$$

RESULTS AND DISCUSSION

The computer program written for the CIM was run, on an IBM 3081G mainframe computer, to investigate the effect of the following: ratios of $\Delta x/\Delta t$, values of the artificial viscosity term θ and the number of nodes N on the pressure transient at point A of the pipeline system shown in Figure 1. In

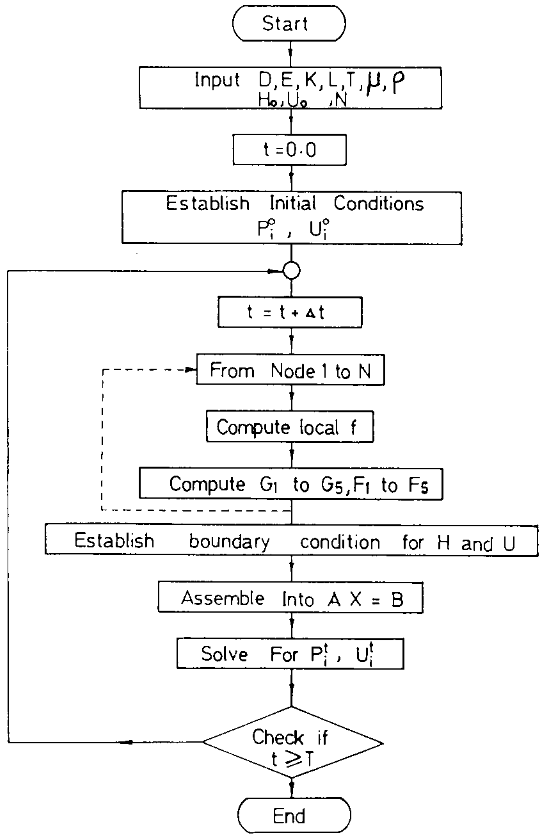

Figure 4. Computer program flow chart for the centre implicit method

this study, the effect of the number of nodes on pressure transient has been considered. If the effect of grid length is required, it can be determined by dividing the pipe length by (N-1). The computer program written for the MOC[3] was also run in the same computer and the CPU times were determined for different numbers of nodes. In both methods of computation, analyses were carried out with and without friction in the pipeline.

Stability criteria

The stability criteria based on the ratios of the linear (Δx) and time (Δt) increments can be best examined by assuming a pipeline system which is frictionless. The pressure transient of such a system would be a square wave and the effect of the artificial viscosity term is not present; equations (5)–(11) are modified in the computer program by putting $\theta = 1$. The results of the investigation of the stability criteria are given in Figure 5.

Figure 5 shows that the stable condition is obtained only when $\Delta x/\Delta t = c$. This results contradicts the earlier finding[3] that CIM has no restriction on the ratio of $\Delta x/\Delta t$. In the same reference, it has been commented that the CIM will yield unsatisfactory results for sudden and sharp transients, but the present study shows that satisfactory results can be obtained if the stability criterion is satisfied.

In practice, the sudden pressure change at a point a will be noticed at a distance Δx away, after a lapse of $\Delta x/c$ seconds, but simultaneous solution of equations (8) and (10) will indicate an

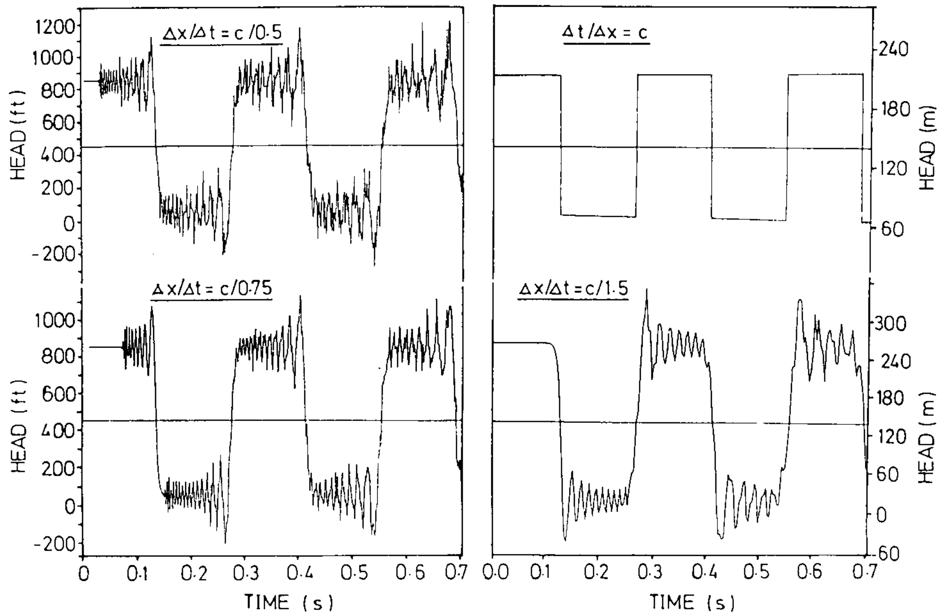

Figure 5. Predicted pressure transients without friction using CIM for different $\Delta x/\Delta t$ ratios when $\theta = 1{\cdot}000$ and $N = 31$

appreciable change in pressure and time at a distance Δx away after a lapse of Δt s. Therefore, $\Delta x/\Delta t = c$ is a necessary stability criterion for the CIM. Another point to note is that the Courant–Friedrich–Lewy (CFL) stability criterion[10,11] for the solution of hyperbolic equations by the MOC is $\Delta t < \Delta x/c$.

Criteria on artificial viscosity (θ)

The criterion $\Delta x/\Delta t = c$ provides unconditionally stable and accurate results, as long as the pipe system is frictionless. If the friction is included for the same case, then the pressure transient oscillates for $\Delta x/\Delta t \neq c$, and an overshoot occurs in the pressure transient versus time curves for $\Delta x/\Delta t = c$, as seen in Figure 6. If the value of friction is changed then similar phenomena are noticed and the characteristics of oscillation vary slightly.

To eliminate the overshoot in the pressure transient at $\Delta x/\Delta t = c$, the artificial viscosity term θ is introduced, together with the derivative of the variable x, which is shown in equation (5). In this study, the theoretical prediction is compared with that of the experimental results presented by Streeter and Lai[3] for a point near the valve. Therefore, the effect of θ and N on the pressure transient at a point near the valve has been investigated. The investigation of any other point is not considered because pressure transients at other points would be lower in magnitude. Figure 7 shows the effect of θ on the wave form of the pressure transient for the case of sudden closure of the valve, when $\Delta x/\Delta t = c$. From this it can be concluded that the value $\theta = 1{\cdot}005$ gives a good wave form for the pressure transient with friction, that the result is very sensitive to small changes in the value of θ and that a change in the pipe friction does not affect this value. Figure 8 shows the effect of the number of nodes on the shape of the curve when $\Delta x/\Delta t = c$ and $\theta = 1{\cdot}005$; as the number of nodes is decreased below 31, the shape of the curve is affected by waviness at the extreme pressure transients; a similar effect is not noticed when the number of nodes is increased beyond 31. The comparison of Figures 7 and 8 shows that the change of θ affects the pressure at extremities of the pipe, whereas the change of N affects the pressure along

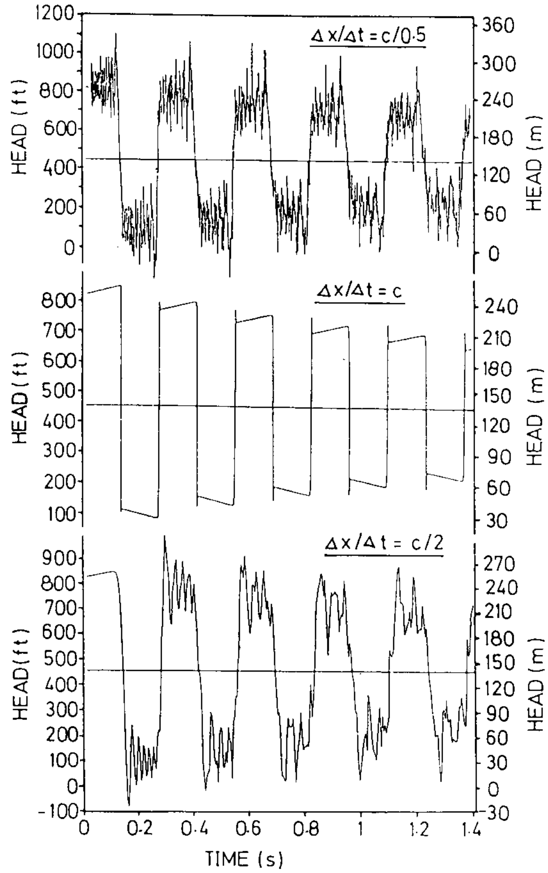

Figure 6. Predicted pressure transients with friction using CIM for different $\Delta x/\Delta t$ ratios when $\theta = 1{\cdot}000$ and $N = 31$

the pipeline, when other variables are kept constant.

Accuracy and computer time

The other important parameters which need attention in the numerical analysis of engineering problems are the accuracy and the computer time. These two parameters were investigated by varying the number of nodes N in the pipeline system. The two variables $\Delta x/\Delta t = c$ and $\theta = 1{\cdot}005$ were kept constant and the numbers of nodes were varied to determine the effect of number of nodes on the accuracy and the computer time.

Figure 8 shows that the accuracy of CIM does not increase very much beyond $N = 31$, i.e. $\Delta x < 3{\cdot}0$ m. However, the computer time, as given in Table I increases with the number of nodes and a compromise should be struck between the accuracy and the computer time. The comparison of computer time for the CIM and the MOC shows that the CIM requires more computer time than the MOC. This is because the CIM involves the solving of simultaneous equations of all unknowns at the next time step, whereas the MOC involves only step-wise iterations.

Comparison with experimental results

The experimental results presented in an earlier work by Streeter and Lai[3] at two Reynolds

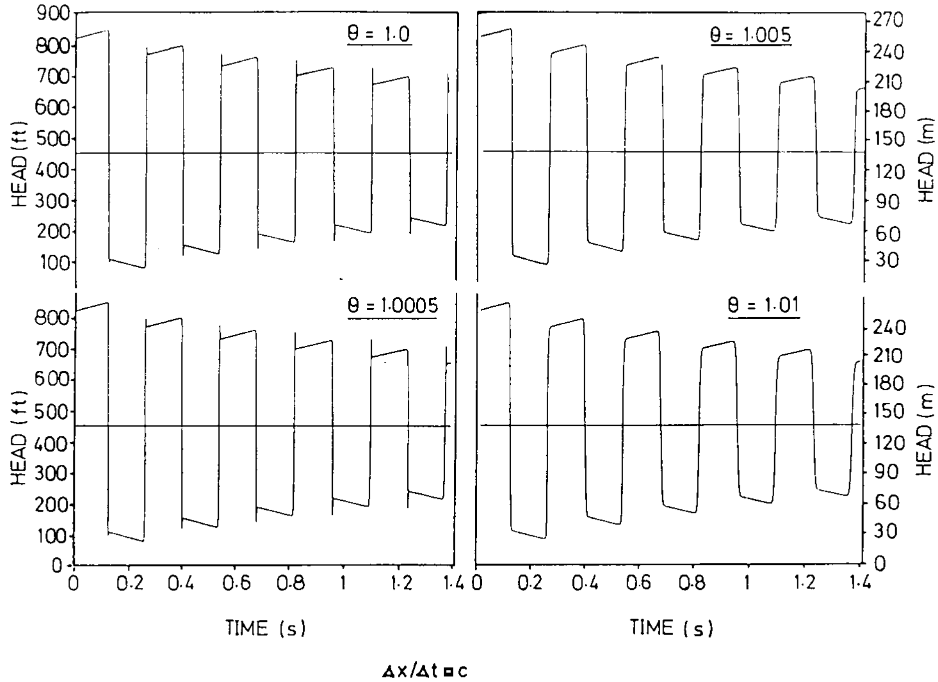

Figure 7. Predicted pressure transients with friction using CIM for different values of node N when $\Delta x/\Delta t = c$ and $\theta = 1{\cdot}005$

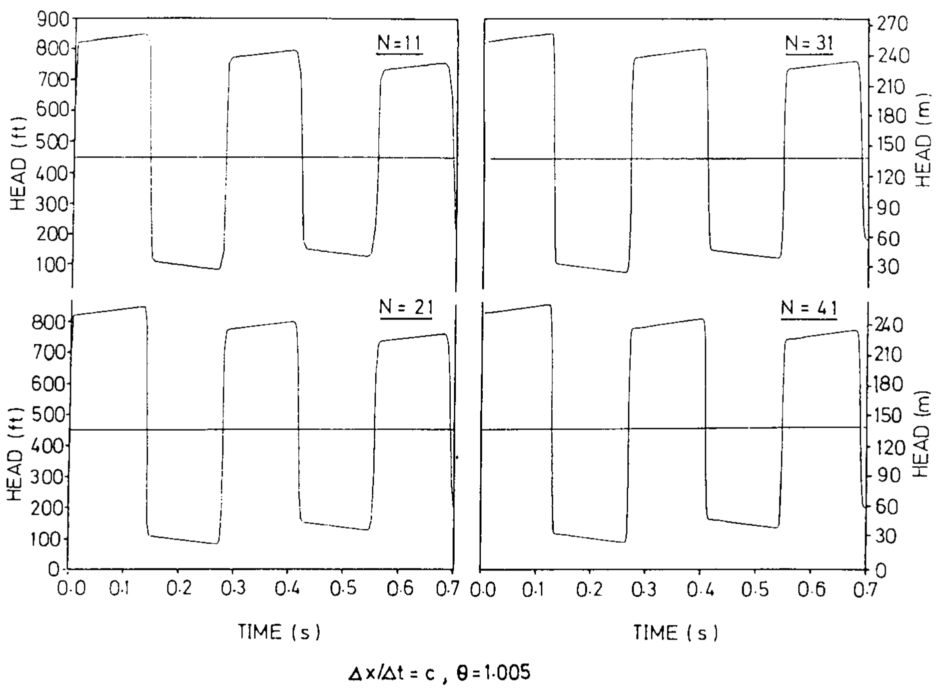

Figure 8. Predicted pressure transients with friction using CIM for different values of θ when $\Delta x/\Delta t = c$ and $N = 31$

Table I. Comparison of CPU times for MOC and CIM; $L = 91{\cdot}47$m

N	CPU time, s		CIM − MOC
	CIM	MOC	MOC
11	2·05	1·75	17·0
21	21·54	18·03	19·0
31	93·57	69·33	35·0
41	246·69	160·48	53·3

Table II.

	Case 1		Case 2	
Bulk modulus K:GPa (kpsi)	2·2774	(33·000)	2·212	(32·000)
Viscosity $\mu \times 10^6$:m^2/s (ft^2/s)	0·6414	(6·904)	0·9160	(9·859)
Density ρ:kg/m^3 (lb/ft^3)	992·8	(61·98)	999·0	(62·37)
Reservoir head H:m(ft)	137·46	(451)	13·72	(45)
Velocity u:m/s (ft/s)	0·896	(2·94)	0·112	(0·367)
Reynolds number	15,330		1·340	

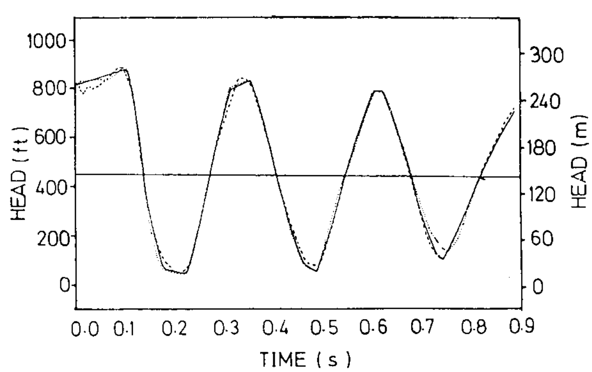

Figure 9. Comparison of experimental and theoretical pressure transients at Reynolds number 15,330: ——— centre implicit method; — — — Streeter's experimental results ; ......, Streeter's MOC

numbers (Table II) were compared with the theoretical results obtained using the proposed method, in which $N = 31, \Delta x/L = 0{\cdot}1, \Delta t = \Delta x/c$ and $\theta = 1{\cdot}005$.

Figure 9 shows the experimental and theoretical results at the Reynolds number 15,330. There is a very close agreement between the results; however, the theoretical prediction with the CIM has a slight edge over the MOC, which may be due to the fact that the CIM is better suited for modelling of the friction than the MOC. The comparison of the results at the Reynolds number 1340, in Figure 10, shows that the results obtained with the CIM and the MOC agree well but both differ from the experimental results. The discrepancy is due to incorrect modelling of the flow in the pipe line at low Reynolds number, which has been eliminated using a two-dimensional model.[8] The results of the two-dimensional model will be published later.

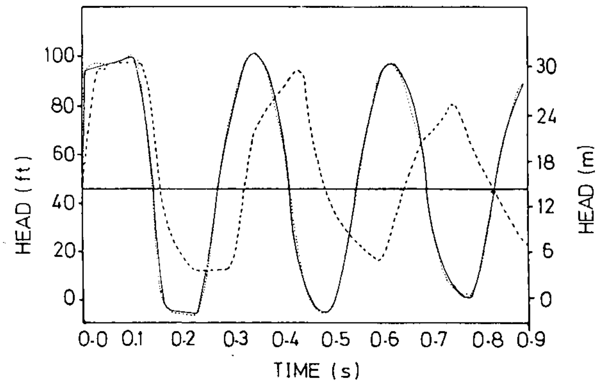

Figure 10. Comparison of experimental and theoretical pressure transients at Reynolds number 1340: ——— centre implicit method; ——— Streeter's experimental results; ....., Streeter's MOC

CONCLUSIONS

It can be concluded that the centre implicit method can be used as an alternative method in a pressure transient study of pipelines instead of the method of characteristics, which is widely used. The centre implicit method used for predicting the pressure transient gives satisfactory results for sudden valve closure, provided that the stability criterion $\Delta x/\Delta t = c$ and the value of the artificial viscosity term $\theta = 1{\cdot}005$ are satisfied. The study shows that the CIM is very sensitive to small changes of the stability criterion $\Delta x/\Delta t$ and the artificial viscosity term θ. For the same number of nodes, the computer time required with the CIM is more than that with the MOC and this is expected because of the imposed stability criterion. It may seem that the proposed CIM has discarded the major advantage of large time step that is normally used in an implicit approach. But for the accurate prediction of the fast pressure transients such as these, the stability criterion rule must be employed at the expense of computer time.

The comparison of pressure transients which are theoretically predicted using the CIM and the MOC, with the experimental results shows that both methods are equally valid for one-dimensional analysis of pressure transients at high Reynolds number. At low Reynolds number, both methods lead to theoretical results which differ from the experimental results, after the first transient step. However, two-dimensional analysis of transient flow leads to better results and, for this purpose, the implicit method is better suited than the method of characteristics; the results will be presented later.

LIST OF SYMBOLS

$\mathbf{A}$	Matrix of coefficients
$\mathbf{B}$	Column of matrix
c	Acoustic velocity
D	Diameter of the pipe
E	Young's modulus of the pipe material
$F_1, \ldots, F_5$	Coefficients
f	Friction factor
$G_1, \ldots, G_5$	Coefficients
g	Acceleration due to gravity
H	Pressure head in height of water column

i	Any node in the x-direction
L	Length of the pipe
N	Number of nodes in the x-direction
p	Average pressure at a cross-section
T	Thickness of the pipe
t	Time
u	Mean axial velocity at a cross-section
x	Horizontal distance along the pipe
z	Elevation of the pipe
Δt	Time increment
Δx	Linear distance increment
θ	Artificial viscosity
μ	Viscosity
ρ	Density of the liquid
ϕ	Variable representing either p or u

REFERENCES

1. R. W. Angus, 'Simple graphical solution for pressure rise in pipes and pump discharge lines', *Journal of Engineering Institute, Canada*, (February), 72–81 (1935).
2. C. A. M. Gray, 'The analysis of the dissipation of energy in waterhammer', *Proceedings of ASCE*, **119,** 1176–1194 (1953).
3. V. L. Streeter and C. Lai, 'Waterhammer analysis including fluid friction', *Proceedings of ASCE*, **88,** (HY2), 79–112 (1962).
4. V. L. Streeter, 'Unsteady flow calculation by a numerical method', *Journal of Basic Engineering, ASME*, (June) **94,** 457–466 (1972).
5. V. L. Streeter and E. B. Wylie, *Hydraulic Transients*, McGraw-Hill, New York, 1967.
6. J. A. Fox, *Hydraulic Analysis of Unsteady Flow in Pipe Networks*, Macmillan Press, London, 1977.
7. D. C. Wiggert and M. J. Sundquist, 'The effect of gaseous cavitation in fluid transients', *Journal of Fluid Engineering, ASME*, **101,** 79–86 (1979).
8. J. K. Tan, 'Study of pressure transients in pipelines', *M. Eng. Thesis*, National University of Singapore, 1986.
9. P. J. Roache, 'On artificial viscosity', *Journal of Computational Physics*, **10** (2), 71–85 (1972).
10. K. C. Ng and M. N. A. Hawlader, 'The prediction of pressure difference in a downcomer pipe during a depressurization process', *Nuclear Energy*, **23** (4), 273–280 (1984).
11. R. Courant, K. Friedrich and H. Lewy, 'On the partial differential equation of mathematical physics', *IBM Journal*, **11,** 215–243 (1967).